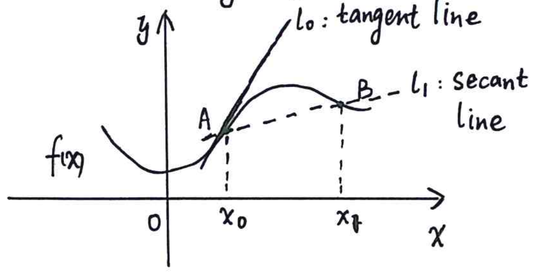
Tutorial notes - 3 whizkiddeng@gmail.com

Geometric meaning of derivative.



Assume we have 2 points A, B on the graph of f(x), connect A, B we get a secant line AB which implies a direction $\vec{AB}$ (from A to B),

But the direction we wanted is the one move along with the curves, which is the tangent line l_0, so from l_1 to l_0 we just need to let B to be closer and closer to A , which means "limit"

So from above discussion, we can get our definition of derivative:

$$f'(x_0) = K_{l_0} \text{ (slope of } l_0) = \lim_{l_1 \to l_0} K_{l_1} = \lim_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1 - x_0} \qquad (1)$$

Remark: For the differentiability at x_0, we must chect the existence of the limit in (1). And the limit contains 2 parts for $x_1 \to x_0$ means

$x_1 \to x_0^+$ ($x_1 > x_0$, from right-hand side)

$x_1 \to x_0^-$ ($x_1 < x_0$, from left-hand side)

Applications of derivative.

Thm-1: If $\begin{cases} f'(x) > 0 \\ f'(x) < 0 \end{cases}$ in D(f) then f(x) is strictly $\begin{cases} \text{increasing} \\ \text{decreasing} \end{cases}$ in D(f).

It can be deduced from definition like:

$\forall$ x_0 in D(f)

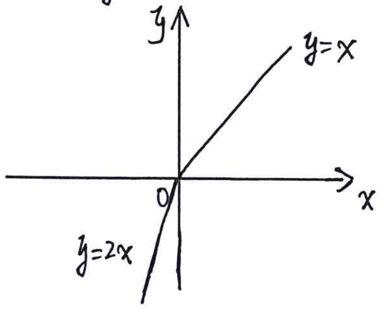
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x - x_0} > 0 \quad \begin{cases} \text{If } x > x_0 \Rightarrow f(x) > f(x_0) \\ \text{If } x < x_0 \Rightarrow f(x) < f(x_0) \end{cases} \Rightarrow \text{strictly increasing}$$

This is not a rigorous proof, just a simple explaination. The proof need use the mean-value thm.

Remark: this thm said $f'(x) > 0 \Rightarrow f(x) \nearrow$

But the converse is not always true, means $f(x) \nearrow \not\Rightarrow f'(x) > 0$

for we may have some points is not differentiable. The example as follows:



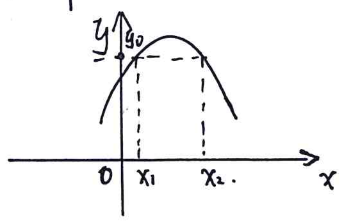
$$f(x) = \begin{cases} x & x \geq 0 \\ 2x & x < 0 \end{cases}$$

of course $f(x) \nearrow$ in R. But $f(x)$ is not differentiable at $x=0$

which implies we don't have $f'(x) > 0$ in R.

Thm-2. If $f(x)$ is strictly $\begin{cases} \text{increasing} \\ \text{decreasing} \end{cases}$ then $f(x)$ has inverse.

An explaination like:



we don't have inverse for such function.

Because if we draw a horizontal line we may have 2 cross points with curves, even more.

which means $f(x_1) = f(x_2) = y_0$, $x_1 \neq x_2$.

So it's contradict from the definition of inverse.

Remark: It's similar that the converse is wrong means:

$f(x)$ has inverse $\not\Rightarrow$ $f(x)$ is $\nearrow$ / $\searrow$, example like discrete function.

$$\begin{cases} f(1) = 2 \\ f(2) = 1 \\ f(3) = 3 \end{cases} \Rightarrow \begin{cases} f^{-1}(2) = 1 \\ f^{-1}(1) = 2 \\ f^{-1}(3) = 3 \end{cases}$$

but $f(x)$ is not $\nearrow$ / $\searrow$.

Q3. Use the definition to compute $f'(0)$

(1) $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x - 0}$$

$$= \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

For $|x\sin\frac{1}{x}| \leq |x|$, so when $x \to 0$, then $|x| \to 0$

which force $|x\sin\frac{1}{x}| \to 0$. this is the "sand-wich" thm

(2) $f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$

similarly $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \sin\frac{1}{x}$

this limit doesn't exist. For if we choose 2 different sequence of x like:

$x_k = \dfrac{1}{2k\pi+\frac{\pi}{2}}$, $k=1,2,3\cdots$ so $k\to+\infty \Rightarrow x_k \to 0$

$y_n = \dfrac{1}{2nk-\frac{\pi}{2}}$, $n=1,2,3\cdots$ so $n\to+\infty \Rightarrow y_n \to 0$

But $\begin{cases} \sin\frac{1}{x_k} = \sin(2k\pi+\frac{\pi}{2}) = 1 \\ \sin\frac{1}{y_n} = \sin(2n\pi-\frac{\pi}{2}) = -1 \end{cases}$ which contradicts from the uniqueness of limit.

so $f'(0)$ doesn't exist, $f(x)$ is not differentiable at $x=0$.

Q4. Assume $f'(x_0)$ exists, try to prove that:

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} = f'(x_0). \qquad (2)$$

Pf: Recall the definition of $f'(x)$:

$f'(x_0) = \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$ If we use the substitution $x = x_0+h$, so $x\to x_0$ just means $h\to 0$

then we get the equivlance definition for $f'(x)$:

$f'(x_0) = \lim\limits_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ (3) of course h can be positive or negative.

so (2) we have to prove is the symmetric form of difference.

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{2h}.$$

$$= \lim_{h\to 0} \frac{1}{2}\left(\frac{f(x_0+h)-f(x_0)}{h} + \frac{f(x_0-h)-f(x_0)}{-h} \right) \qquad (4)$$

both parts have the same form with (3), so from the assumption that $f'(x_0)$ exists. we have:

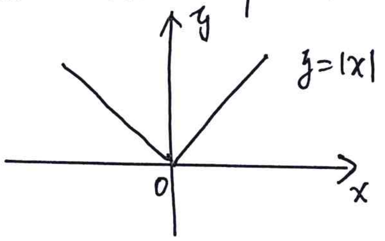
$(4) = \frac{1}{2}\left(f'(x_0)+f'(x_0)\right) = f'(x_0)$ done.

Remark: This problem tells: $f'(x_0)$ exists $\Longrightarrow \lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$ exists and equals to $f'(x_0)$

But the converse is not true. means:

$\lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$ exists $\not\Longrightarrow$ $f'(x_0)$ exists.

The counter-example like:



just $y=|x|$ at $x_0=0$.

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} = \lim_{h\to 0} \frac{f(h)-f(-h)}{2h} = \lim_{h\to 0} \frac{|h|-\overset{\overset{|h|}{\parallel}}{|-h|}}{2h} = 0 \text{ exists.}$$

But we know $f(x)=|x|$ is not differentiable at $x=0$.

A more advanced example is the Dirichlet function which is defined like:

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is irrational number} \\ 0 & \text{when } x \text{ is a rational number.} \end{cases}$$

for $y=|x|$, we just have the contradictory in $x=0$. but for Dirichlet function

we have $\lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} = 0$ in every point in real line R.

But $f(x)$ is dis-continuous in R so it's not differentiable.

If you are interesting in it, you can google for "Dirichlet function" for more.

Some notes about the injective and surjective.

① Function $f(x): A \to B$. (A, B are the subsets of R)

A: domain of the $f(x)$, notice may not the maximal domain of $f(x)$, means: $\boxed{A \subseteq \text{max-domain}(f)}$

Ex: $f(x) = |x-2| + 3$. max-domain$(f) = R$ (whole real line)

But we can define $f(x)$ just in ~~R^+~~ $A = R^+$ (positive real number)

then $A \subsetneqq$ max-domain (f).

B: what is called co-domain. can be larger than the $f(A)$ (range of $f(x)$ on set A)

actually the surjective is just to check whether $\boxed{B = f(A)}$ or not.

② Injective.

First recall the definition: Injective means "one-to-one", that given any two points $x_1, x_2 \in A$ $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$, simplely says, different points have different values.

i) so it's very easy to check $f(x)$ is not injective if we can find two different points $x_1 \neq x_2$, satisfied $f(x_1) = f(x_2)$, just like the example above:

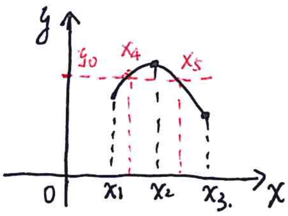
$f(1) = |1-2|+3 = 4 = |3-2|+3 = f(3)$, so $f(x)$ is not injective.

ii) then how to show $f(x)$ is injective? For the $\boxed{\text{continuous}}$ function, we have an useful conclusion.

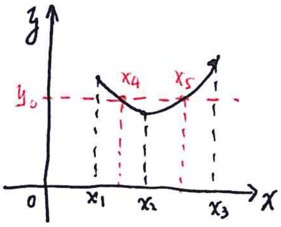
Thm: If $f(x)$ is $\boxed{\text{continuous}}$ in $[a, b]$, then $f(x)$ is $\boxed{\text{injective}}$ in $[a, b] \Longleftrightarrow f(x)$ is $\boxed{\text{increasing}}$ or $\boxed{\text{decreasing}}$ in $[a$

The proof of this thm need use the $\boxed{\text{intermediate value theorem}}$ which you would learn later, so just leave it as a simple exercise later. Here we just give some explainations.

If $f(x)$ is neither increasing nor decreasing in $[a, b]$, then it must acts like:



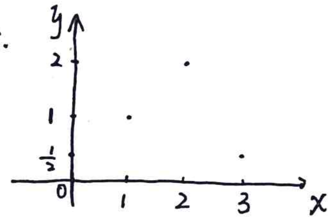
or



so $f(x)$ is not injective.

Notice that the $\boxed{\text{continuous}}$ is necessary. If f is not continuous, then the thm no longer holds.

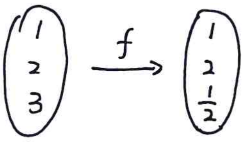
Ex.



$f(1) = 1$

$f(2) = 2$

$f(3) = \frac{1}{2}$,



f is a discrete function, clearly $f(x)$ neither increasing not decreasing. but $f(x)$ is injective indeed.

So according to the above thm, when we meet continuous functions $f(x)$ in $[a,b]$ (it includes most cases) we just have to check whether $f(x)$ is increasing / decreasing or not.

③. Surjective.

Def: Given any $y_0 \in B$. if we can always find at least one $x_0 \in A$, s.t $f(x_0) = y_0$. then $f(x)$ is surjective.

So from the definition we can see the process to show surjective acts like "solving equation". But now the y_0 becomes known, x_0 is unknows, we try to solve a x_0 in the form of y_0.

Ex. $f(x) = x^2 - 1$. $A = \text{max-domain}(f) = R$, B to be decided later.

in order to check the surjective, take any $y_0 \in B$. solve $f(x_0) = y_0$. just:

$$x_0^2 - 1 = y_0 \Rightarrow x_0^2 = y_0 + 1. \quad (1)$$

so in order to make sure (1) has solutions $x_0 \in R$. we must have $y_0 + 1 \geq 0 \Rightarrow y_0 \geq -1$.

then we have different cases:

i). $B = [-1, +\infty)$, any $y_0 \in B$ satisfied $y_0 \geq -1$, then (1) always has solution.

so $f(x)$ is surjective.

ii) $B = [-2, +\infty)$, $y_0 \in B$ implies y_0 may lies in $[-2, -1)$ means $-2 \leq y_0 < -1$,

in such case $y_0 + 1 < 0$, then (1) doesn't have solution, that's means for $-2 \leq y_0 < -1$, we can't find any $x_0 \in A = R$, s.t $f(x_0) = y_0$.

so $f(x)$ is not surjective.

That's how the co-domain B would change the surjective of $f(x)$. so we need pay attention to the B.

Math1010 Tutorial 3 (Prepared by Chung Shun Wai)

Topics: definition of derivatives of functions with applications.

Q1) By using definition of derivatives, find the first derivative of $f(x)$ (i.e. $f'(x)$) for any $x \in \text{Domain}(f)$.

a) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = 3x - 2$

b) $f : \mathbb{R}^{+} \to \mathbb{R}$; $f(x) = \sqrt{x+1}$

c) $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \sqrt{x + \sqrt{x}}$

Q2) Sketch the graph $y = f(x)$, and by using the definition of derivatives, find the slope of the graph at each $x \in \mathbb{R}$.

a) $y = f(x) = (x+1)^2$

b) $y = f(x) = e^x$

c) $y = f(x) = \sin x$

Recall :

- definition of derivative of a function :

$$f'(x) \overset{\text{def}}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Slope of the graph $y = f(x)$:

Slope at point x $= f'(x)$

Solution

Q1a) given $f(x) = 3x - 2$,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)-2]-[3x-2]}{h}$$

$$= \lim_{h\to 0} \frac{3h}{h} = 3$$

Q1b) given $f(x) = \sqrt{x+1}$, for any $x \in \text{Domain}(f) = \mathbb{R}^+$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

Q1c) given $f(x) = \sqrt{x+\sqrt{x}}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{2} = \lim_{h \to 0} \frac{\sqrt{x+h+\sqrt{x+h}} - \sqrt{x+\sqrt{x}}}{h}.$$

$$= \lim_{h \to 0} \frac{(x+h+\sqrt{x+h}) - (x+\sqrt{x})}{h\sqrt{x+h+\sqrt{x+h}} + \sqrt{x+\sqrt{x}}}$$

$$= \lim_{h \to 0} \left[\frac{h}{h(\sqrt{x+h+\sqrt{x+h}} + \sqrt{x+\sqrt{x}})} + \frac{(x+h)-(x)}{h(\sqrt{x+h+\sqrt{x+h}} + \sqrt{x+\sqrt{x}})(\sqrt{x+h}+\sqrt{x})} \right]$$

$$= \lim_{h \to 0} \left[\frac{1}{\sqrt{x+h+\sqrt{x+h}} + \sqrt{x+\sqrt{x}}} + \frac{1}{(\sqrt{x+h+\sqrt{x+h}} + \sqrt{x+\sqrt{x}})(\sqrt{x+h}+\sqrt{x})} \right]$$

$$= \frac{1}{2\sqrt{x+\sqrt{x}}} + \frac{1}{4\sqrt{x}\sqrt{x+\sqrt{x}}} = \frac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}$$

//

Q2a) given $y = f(x) = (x+1)^2$

Slope at $x = f'(x)$

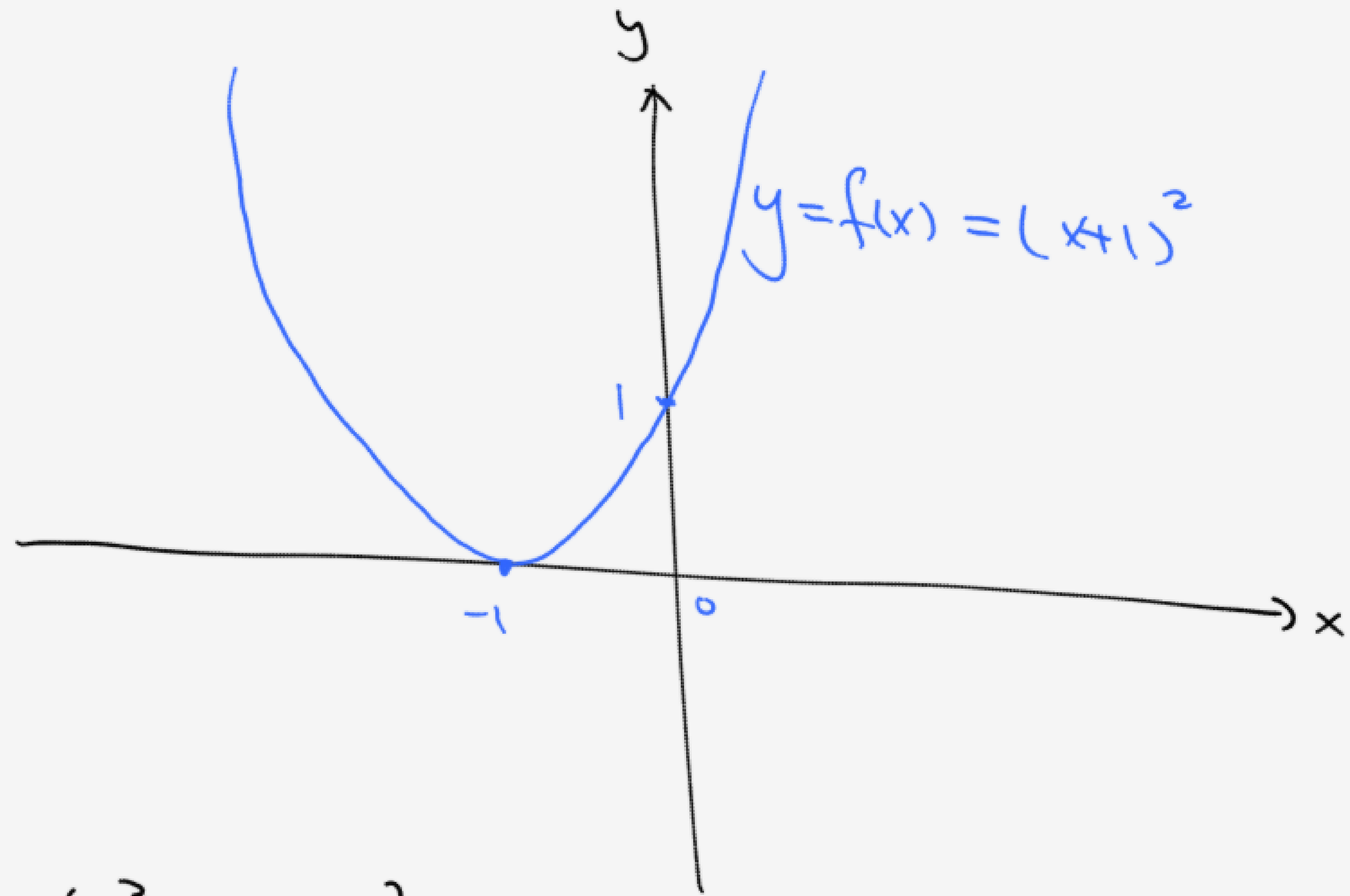
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h+1)^2 - (x+1)^2}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 + h^2 + 1 + 2xh + 2x + 2h) - (x^2 + 2x + 1)^2}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2xh + 2h}{h}$$

$$= \lim_{h \to 0} h + 2x + 2 = 2x + 2 \;//$$



Q2b) given $y = f(x) = e^x$

Slope at $x = f'(x)$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

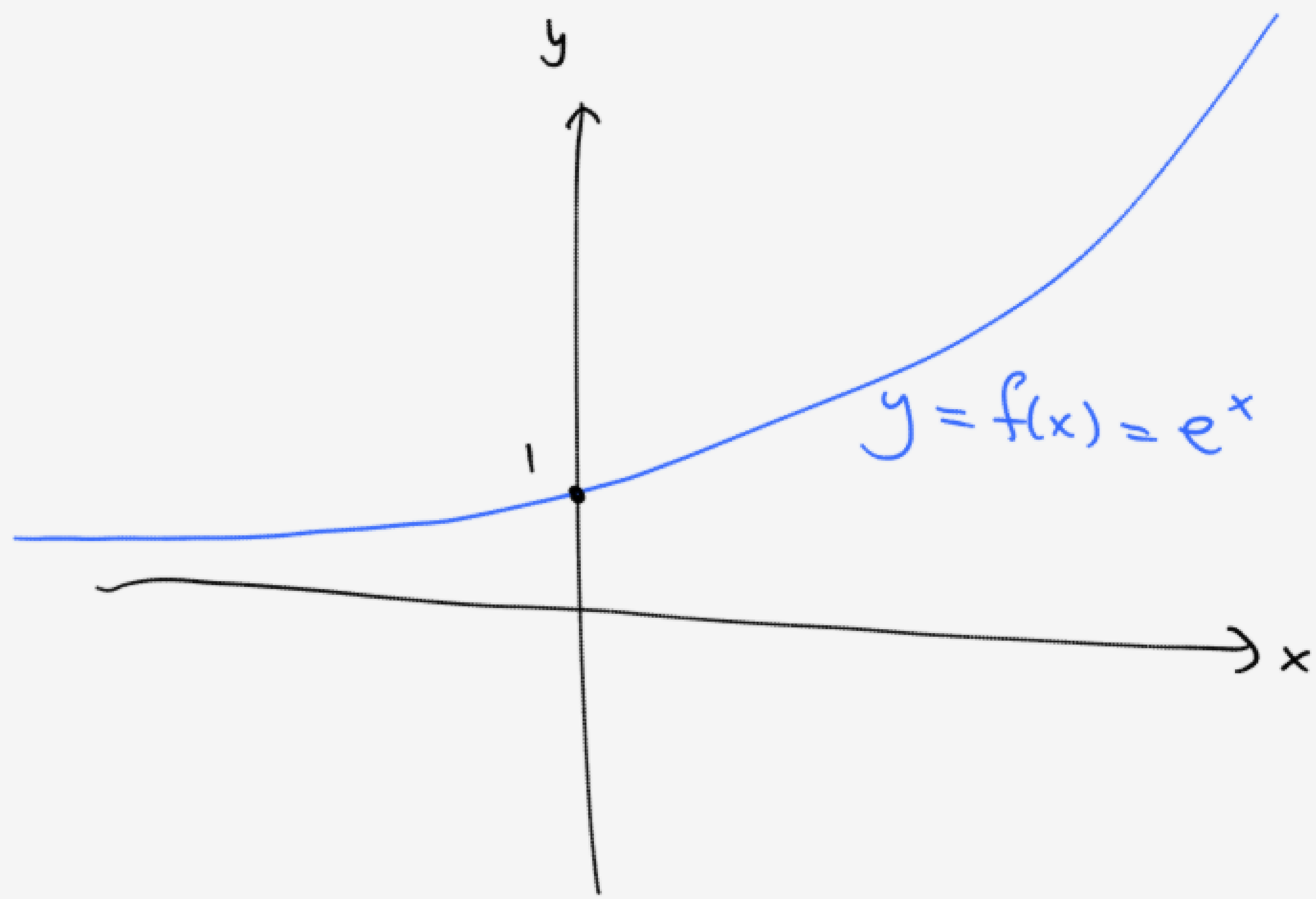
$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x}{h}(e^h - 1)$$

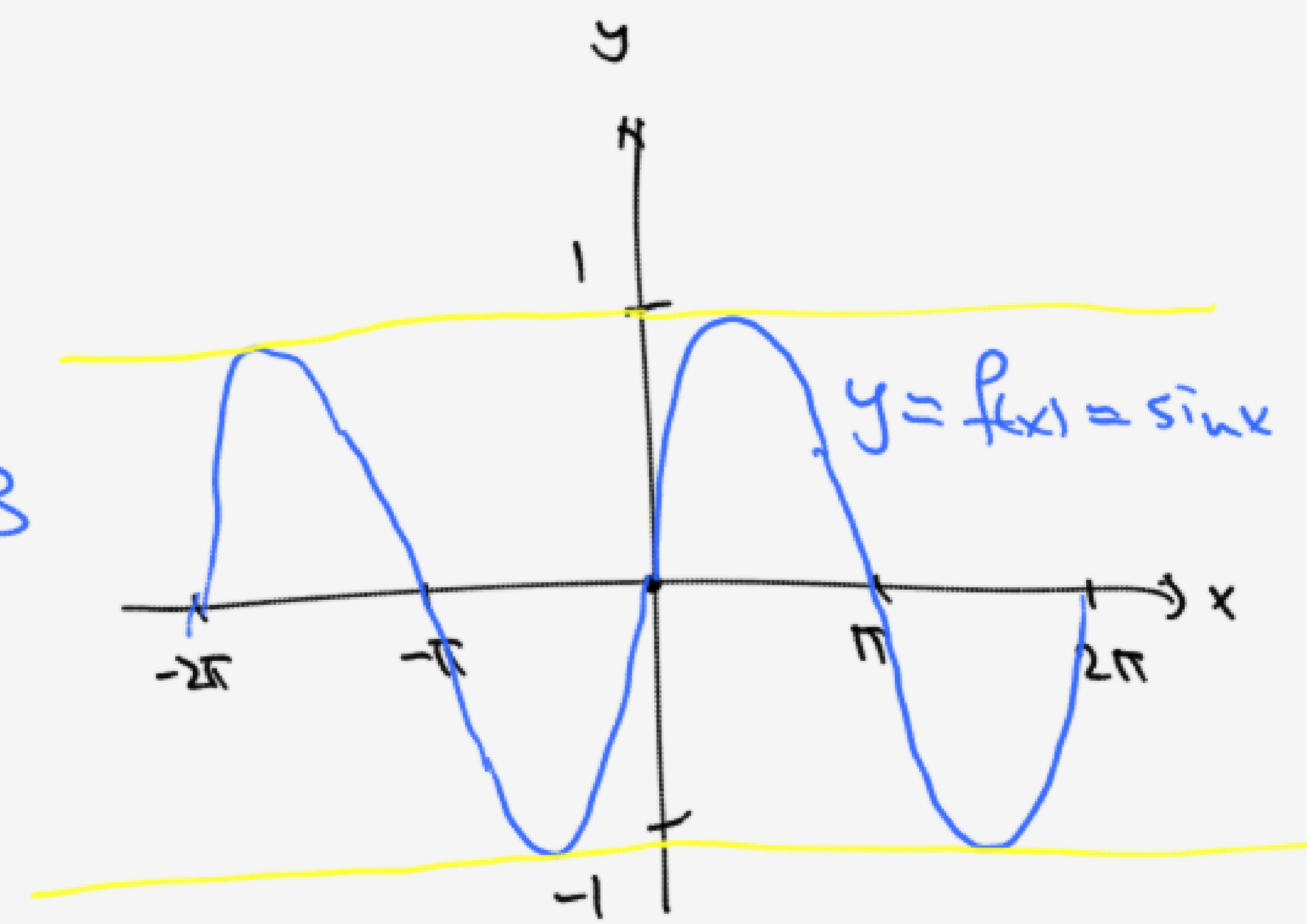
$$= \lim_{h \to 0} \frac{e^x}{h}\left(\frac{h^1}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots \right)$$

$$= \lim_{h \to 0} e^x\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \cdots \right)$$

$$= e^x \quad //$$



2c) given that $y = f(x) = \sin x$



Recall that
$$\begin{cases} \sin(A+B) = \sin A \cos B + \cos A \sin B \\ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \end{cases}$$

Slope at x $= f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\sin(x+h) - \sin x \right)$

$$= \lim_{h\to 0} \frac{1}{h}\left(\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x) \right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\sin(x)\left(\cos(h) - 1\right) + \cos(x)\sin(h) \right]$$

$$= \lim_{h\to 0}\left[\sin(x)\left[\frac{1}{h}\left(-\frac{h^2}{2!} + \frac{h^4}{4!} + \cdots\right)\right] + \cos(x)\left[\frac{1}{h}\left(h - \frac{h^3}{3!} + \frac{h^5}{5!} + \cdots\right)\right]\right]$$

$$= \cos(x)$$

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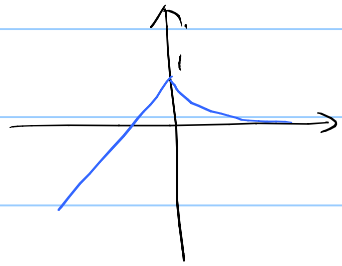
* Left and right limit

Thm: $\lim\limits_{x\to a} f(x)$ exists $\Leftrightarrow$ $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ exist.

and $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x)$.

(1) $f(x) = \begin{cases} x+1, & x \le 0 \\ \dfrac{1}{x+1}, & x > 0 \end{cases}$

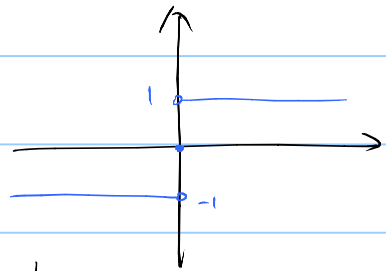


$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x+1) = 1$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{1}{x+1} = 1$

$\Rightarrow \lim\limits_{x\to 0} f(x) = 1$ (exists)

(2) $f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

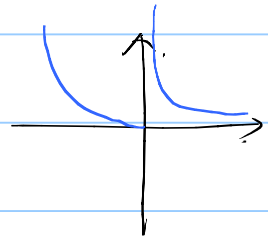


$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{x}{-x} = \lim\limits_{x\to 0^-} (-1) = -1$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{x}{x} = \lim\limits_{x\to 0^+} 1 = 1$

$\Rightarrow \lim\limits_{x\to 0} f(x)$ doesn't exist.

(3) $f(x) = \begin{cases} x^2, & x \le 0 \\ \dfrac{1}{x}, & x > 0 \end{cases}$



$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x^2 = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{1}{x}$ doesn't exist

$\Rightarrow \lim\limits_{x\to 0} f(x)$ doesn't exist.

* Definition of derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{if it exists.}$$

(1) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h} \cdot \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(2) $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos(x+\frac{h}{2})\sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} \cos(x+\tfrac{h}{2}) \frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x$$

Since $\lim_{h\to 0} \cos(x+\frac{h}{2}) = \cos x$

$$\lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = 1$$

→ <u>Application</u> : Marginal cost.

Cost function: $F(x)$, cost $F(x)$ depend on production x

Marginal cost : $f(x) = F'(x)$

Average cost : $g(X) = \frac{F(X)}{x}$

Break-even pt : x when $f(x) = g(x)$

Say. $F(x) = x^3 - 6x^2 + 20x$

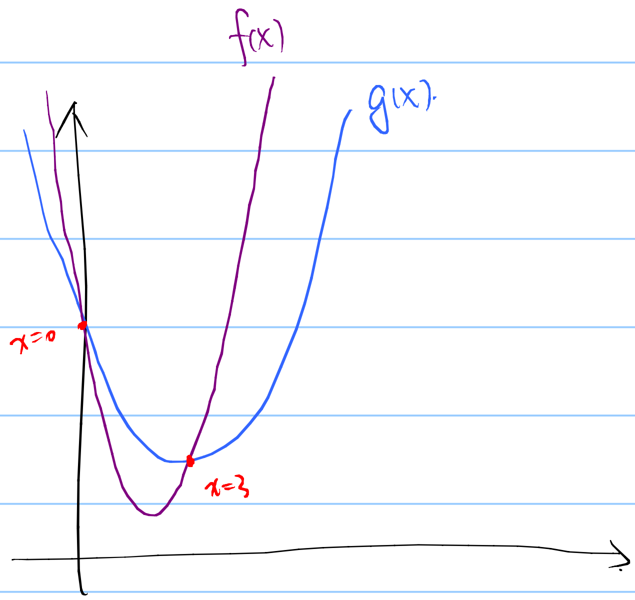
$f(x) = F'(x) = 3x^2 - 12x + 20$

$g(x) = \frac{F(x)}{x} = x^2 - 6x + 20$

$f(x) = g(x) \iff 3x^2 - 12x + 20 = x^2 - 6x + 20$

$\iff 2x^2 - 6x = 0$

$\iff x = 0$ or $x = 3$



Notice that when $f(x) = g(x)$, $g(x)$ is minimized.

Generally, break-even pt is minimum or maximum of $g(x)$

In math, it is called the critical pt of $g(x)$, which means x when $g'(x) = 0$.

We can prove that x is critical point of $g(x) \iff f(x) = g(x)$

Ex: (1) Calculate $\lim_{x \to 0} \frac{\sqrt{1+x^2}-1}{x \sin x}$

(2) derivative of $f(x) = \sqrt{x^2+2}$ using definition.